Integrali doppi impropri

Es a

$$\int_D (x^2+y^2)^{-\alpha}\, dx\, dy \qquad D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\geq 1\}$$

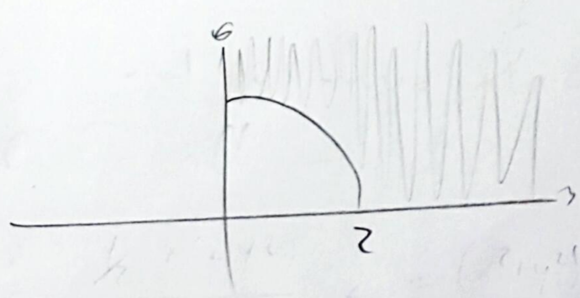
$$\iint_D \frac{1}{(x^2+y^2)^2}$$

$$\frac{1}{(x^2+y^2)^2} = \frac{1}{\left(\sqrt{(x^2+y^2)}\right)^{2\alpha}} \Rightarrow \text{Converge se}$$

$$2\alpha > 2 \Rightarrow \boxed{\alpha > 1}$$

Es b

$$\int \frac{xy}{x^4+y^4}\, dx\, dy \qquad D=\{(x,y)\in\mathbb{R}^2 : x^2+y^2\geq 4 \;\; x\geq 0 \; y\geq 0\}$$



Posso considerare $D_n = \{(x,y)\in\mathbb{R}^2 : 2 \leq x^2+y^2 \leq n\}$

$$f_n = \frac{xy}{x^4+y^4}\chi_{D_n}$$

$f_n \to f$ monotonamente $\Rightarrow$ Beppo Levi

$$\lim_{n\to+\infty} \iint_{D_n} \frac{xy}{x^4+y^4}\,dx\,dy =$$

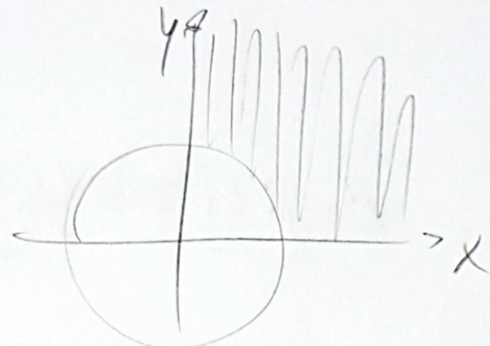
$$\lim_{n\to+\infty} \int_0^{\frac{\pi}{2}} \int_2^{n} \frac{\rho^3 \cos\theta \sin\theta}{\rho^4\cos^4\theta+\rho^4\sin^4\theta}\,d\rho\,d\theta$$

$$\lim_{n\to+\infty} \int_0^{\frac{\pi}{2}} \frac{\cos\theta\sin\theta}{\cos^4\theta+\sin^4\theta}\,d\theta \cdot \ln\rho\Big|_2^{n} \to +\infty$$

Diverge positivamente

Ese

$$\int \frac{x}{(x^2+y^2)^2}\,dx\,dy \qquad D=\left\{(x,y)\in\mathbb{R}^2:\ x^2+y^2\geq 1,\ x\geq 0\ y\geq 0\right\}$$



$$D_n=\left\{(x,y)\in\mathbb{R}^2:\ 1\leq x^2+y^2\leq n,\ x\geq 0,\ y\geq 0\right\}$$

$f_n \to f$ in D (monotonie)

$\parallel$ $\Downarrow$

$\frac{x}{(x^2+y^2)^2}\chi_{D_n}$ Beppo Levi

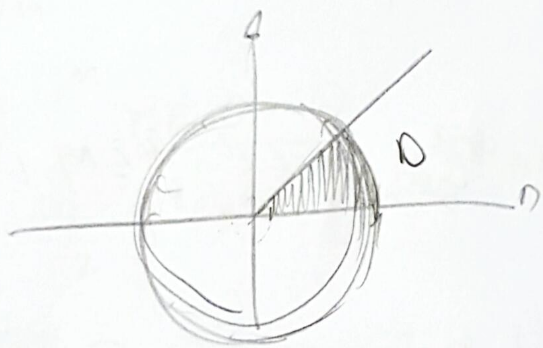
$$\lim_{n\to+\infty}\iint\frac{x}{(x^2+y^2)^2}\,dx\,dy=\lim_{n\to+\infty}\int_0^{\frac{\pi}{2}}\int_1^n\frac{\rho\cos\theta}{\rho^{4/2}}\,d\rho\,d\theta$$

$$\lim_{n\to+\infty}\sin\theta\Big|_0^{\frac{\pi}{2}}\cdot\left(-\frac{1}{\rho}\right)\Big|_1^n=\lim_{n\to+\infty}1\cdot\left(-\frac{1}{n}+1\right)=1$$

Es 6

$$\int_D \frac{x}{(x^2+y^2)}\,dx\,dy$$

$$D = \left\{ (x,y) \in \mathbb{R}^2 : \; x^2+y^2 \le 1 \;\; 0 \le y \le x \right\}$$



Ho problemi in $(0,0)$

Devo verificare la sommabilità

$$\left| \frac{x}{x^2+y^2} \right| \le \frac{1}{x^2+y^2} \le \frac{1}{\sqrt{x^2+y^2}}$$

non posso maggiorare con la radice

$\alpha < 2 \Rightarrow$ sommabile

$$\left| \frac{x}{x^2+y^2} \right| = \frac{|\rho\cos\theta|}{\rho^2} \le \frac{1}{\rho}|\cos\theta| \le \frac{1}{\rho} = \frac{1}{\sqrt{x^2+y^2}}$$

$$|x| = \sqrt{x^2} \le \sqrt{x^2+y^2} \Rightarrow \frac{|x|}{x^2+y^2} \le \frac{1}{\sqrt{x^2+y^2}}$$

$$D_{\frac{1}{n}} = \left\{ (x,y) \in \mathbb{R}^2 : \frac{1}{n^2} \leq x^2+y^2 \leq 1,\ x \geq 0,\ y \geq 0 \right\}$$

$$\frac{1}{n} \leq \rho \leq 1 \qquad \theta \in \left[0, \frac{\pi}{4}\right]$$

$$f_n = f \chi_{D_{\frac{1}{n}}} \to f$$

Per la convergenza dominata passo al limite

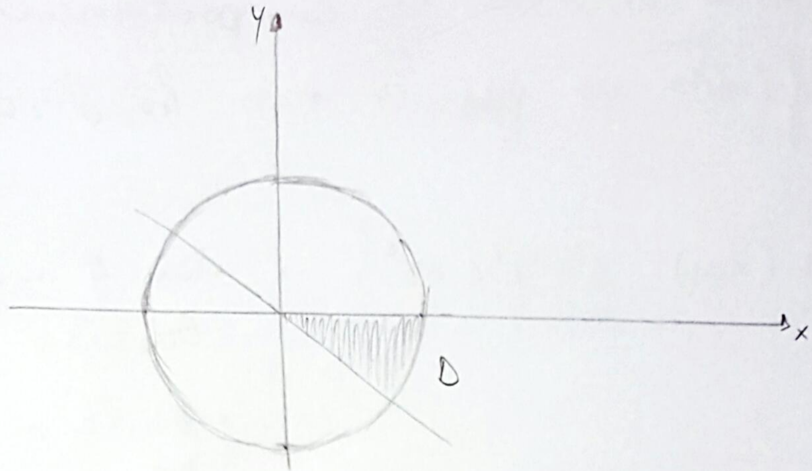
$$\lim_{n \to +\infty} \int_0^{\pi/4} \int_{1/n}^{1} \frac{\rho^2 \cos\theta}{\rho^2} \, d\rho \, d\theta$$

$$\lim_{n \to +\infty} \frac{\sqrt{2}}{2} \left(1 - \frac{1}{n} \right) = \frac{\sqrt{2}}{2}$$

Es e

1

$$\int_D \frac{x}{(x^2+y^2)^2}\,dx\,dy \qquad D=\left\{(x,y)\in\mathbb{R}^2:\ x^2+y^2\le 1,\ -x\le y\le 0\right\}$$



Ho problemi in (0,0)

Verifico la sommabilità:

$$\left|\frac{x}{(x^2+y^2)^2}\right| \le \frac{1}{\left(\sqrt{x^2+y^2}\right)^4} \qquad 4>2 \quad \text{non è sommabile}$$

$\underline{Es}$

$$f(x,y) = e^{-(x^2+y^2)} \qquad (x,y) \in \mathbb{R}^2$$

Devo solo studiare il comportamento all'infinito in quanto non ho discontinuità:

$D_n = \{ (x,y) : x^2+y^2 \leq n^2 \} \rightarrow$
- D_n è monotona crescente per inclusione
- Esiste $n \in \mathbb{N}$ t.c. $\cancel{K}$ è contenuto in D_n (dove K è un insieme chiuso e limitato)

$$\lim_{n \to +\infty} \int_0^{2\pi} \int_0^n \rho e^{-\rho^2} \, d\rho \, d\theta = \cancel{2\pi \left[-\rho e^{-\rho} - e^{-\rho} \right]_0^n} = 2\pi \left(-\frac{e^{-\rho^2}}{2}\right)\Big|_0^n$$

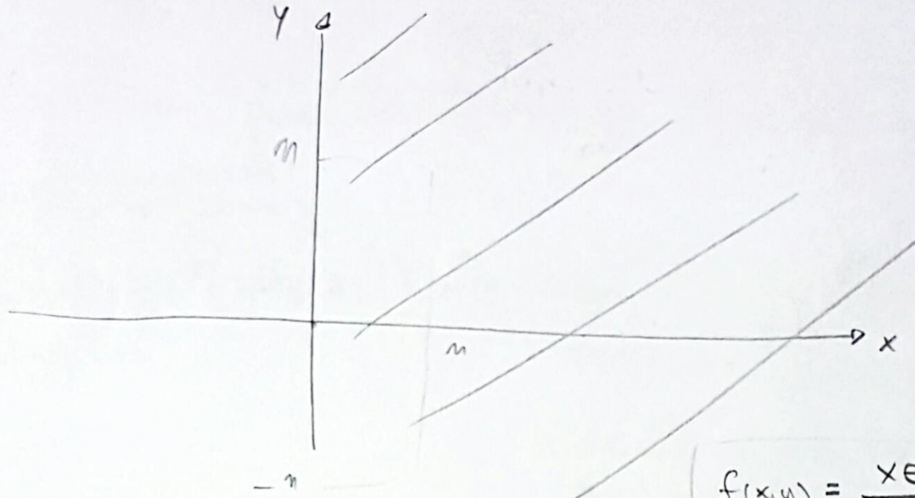
$$= \pi \left(-e^{-n^2} + 1\right) \downarrow \pi$$

$$= \cancel{2\pi \left[-n e^{-n} - e^{-n} + 1 \right]} =$$

Es 1

(3)

$I = \int_\Omega \frac{x e^{-2x^2}}{1+4y^2}\,dx\,dy \qquad \Omega = \{(x,y)\in\mathbb{R}^2 ;\ x\geq 0\}$



$x \in [0,+\infty) \qquad y\in(-\infty,+\infty)$

$f(x,y) = \frac{xe^{-2x^2}}{1+4y^2} = xe^{-2x^2}\,\frac{1}{1+4y^2}$

$D_n: \quad x\in[0,n] \quad y\in[-n,n]$

$f_n \to f$ in D

$f_n = \frac{x e^{-2x^2}}{1+4y^2}\chi_{D_n}$

Tendendo f_n monotonamente a f posso applicare Beppo Levi

$\lim_{n\to+\infty} \int_{-n}^{n} \frac{1}{1+4y^2}dy \int_0^n x e^{-2x^2}dx = \frac{\operatorname{arctg} 2y}{2}\Big|_{-n}^{n} \cdot \frac{e^{-2x^2}}{-4}\Big|_0^n$

$$-\frac{1}{\cancel{8}_{4}}\Big(\underbrace{\operatorname{arctg}(2n)+\operatorname{arctg}(2n)}_{\substack{2\operatorname{arctg}2n\\ \downarrow\\ \frac{\pi}{2}}}\Big)\Big(\underset{\substack{\downarrow\\0}}{e^{-2n^2}}-1\Big) = \frac{\pi}{8}$$

(4)

$$\int_a^{+\infty} e^{\alpha r} < +\infty \;(\text{con } \alpha < 0)$$

__Es 2__

$$\int_0^{+\infty} r^2 e^{-r^2}\, dr$$

$r \cdot re^{-r^2}$

per r grande

perché $r^2 e^{-r^2/2} \le 1$ per r grande

$$r^2 e^{-r^2} \le e^{-r^2/2} \underset{\substack{\uparrow\\ \text{per r grande}}}{\le} e^{-r}$$

$$\lim_{n\to+\infty}\int_0^n r^2 e^{-r^2}\, dr = \cancel{r}\frac{e^{-r^2}}{-2}\Big|_0^n + \frac{1}{2}\int_0^n e^{-r^2} dr$$

$$\lim_{n\to+\infty}\left(\frac{e^{-r^2}}{-2}\Big|_0^n + \frac{1}{2}\int_0^n e^{-r^2} dr\right)$$

$$\lim_{n\to+\infty}\left(\frac{e^{-n^2}}{-2} + \frac{1}{2} + \frac{1}{2}\int_0^n e^{-r^2} dr\right)$$

$$+\frac{1}{2} + \frac{1}{2}\sqrt{\pi}$$

Es 3 (5)

$$I = \int_{\mathbb{R}^2} \left(x + \sqrt{x^2+y^2}\right) e^{-x^2-y^2}\, dx\, dy$$

$$I = \int_0^{2\pi} \int_0^{\infty} (\rho \cos\theta + \rho) e^{-\rho^2} \cdot \rho \, d\rho\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\infty} (\cos\theta + 1)\, \rho^2 e^{-\rho^2}\, d\rho\, d\theta$$

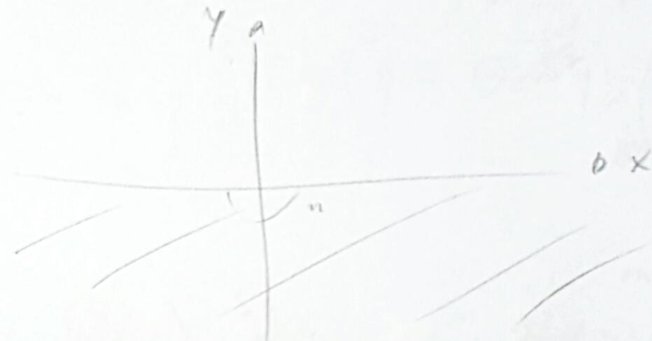
$$\underbrace{-\sin\theta + \theta \Big|_0^{2\pi}}_{= 2\pi} \cdot \left(\frac{1}{2} + \frac{1}{2}\sqrt{\pi}\right) = 2\pi \left(\frac{1}{2} + \frac{1}{2}\sqrt{\pi}\right)$$

$$= \pi + \pi^{\frac{3}{2}}$$

Es 4

$$I = \int_\Omega \frac{y}{(1+x^2+y^2)^2}\, dx\, dy$$

$$\Omega = \{ (x,y) \in \mathbb{R}^2 : y \le 0 \}$$



$$D_n = \{ \cancel{x}(x,y) \in \mathbb{R}^2 : \; 0 \le x^2+y^2 \le n^2 \}$$

$$\lim_{n \to +\infty} \int_\pi^{2\pi} \int_0^n \frac{\rho^2 \sin\theta}{(1+\rho^2)^2}\, d\rho\, d\theta$$

$$\lim_{n \to +\infty} \left[-\cos\theta \Big|_\pi^{2\pi} \cdot \int_0^n \frac{\rho^2}{(1+\rho^2)^2}\, d\rho \right]$$

$$(-1-1) \cdot \left[\int_0^n \frac{\rho^2+1}{(1+\rho^2)^2} - \int_0^n \frac{1\, d\rho}{(1+\rho^2)^2} \right]$$

(7)

$$\lim_{n \to +\infty} -2\arctan n + 2\int_0^n \frac{1}{(1+\rho^2)^2}\, d\rho$$

$$-\pi + 2\lim_{n \to +\infty} \int_0^n \frac{1}{(1+\rho^2)^2} =$$

$$\frac{1}{(1+\rho^2)^2} = \frac{A+B\rho}{(1+\rho^2)} + \frac{C+D\rho}{(1+\rho^2)^2}$$

$$A + A\rho^2 + B\rho + B\rho^3 + C + D\rho = 1$$

$$\int \frac{1}{(1+\rho^2)^2} = \frac{\rho(\rho^2+1)^{-1}}{2} + \frac{1}{2}\arctan\rho \Big|_0^n$$

$$\frac{n}{2(n^2+1)} + \frac{1}{2}\arctan n$$

$$\lim_{n \to +\infty} -2\arctan n + \frac{n}{n^2+1} + \arctan n = -\pi + \frac{\pi}{2} = -\frac{\pi}{2}$$